Math 522 Exam 13 Solutions

1. Let $a, b, c \in \mathbb{Z}$. Suppose that b, c are odd and positive. Prove that $(\frac{a}{bc}) = (\frac{a}{b})(\frac{a}{c})$.

   We factor $b = p_1 p_2 \cdots p_k$ into (not necessarily distinct) odd primes. Similarly, we factor $c = q_1 q_2 \cdots q_m$. Now $bc = p_1 p_2 \cdots p_k q_1 q_2 \cdots q_m$. By the definition of the Jacobi symbol, $(\frac{a}{bc}) = (\frac{a}{p_1})(\frac{a}{p_2}) \cdots (\frac{a}{p_k})(\frac{a}{q_1})(\frac{a}{q_2}) \cdots (\frac{a}{q_m})$. We divide this expression in the natural place to get $(\frac{a}{b})$ and $(\frac{a}{c})$.

2. Calculate $\left(\frac{10}{2021}\right)$ and $10^{1010} \pmod{2021}$. What conclusions, if any, does the Solovay-Strassen test allow us to draw?

   We calculate $(\frac{10}{2021}) = (\frac{2}{2021})(\frac{5}{2021}) = (-1)^{\frac{2021^2-1}{8}}(\frac{2021}{5})(-1)^{\frac{5-1}{2}\frac{2021-1}{2}} = (-1)^{510555}(\frac{1}{5})(-1)^{2020} = -(\frac{1}{5}) = -1$.
   We build up to 10^{1010} gradually; many ways are possible. If your calculator can't handle numbers this big, then you'll need more steps.
   $10^{10} \pmod{2021} = 1055$.
   $10^{100} \equiv (10^{10})^{10} \equiv (1055)^{10} \equiv 74$.
   $10^{1000} \equiv (10^{100})^{10} \equiv (74)^{10} \equiv 683$.
   $10^{1010} = 10^{1000}10^{10} \equiv 683 \cdot 1055 \equiv 1089$.

   Since $-1 \not\equiv 1089 \pmod{2021}$, the Solovay-Strassen test allows us to conclude that 10 is a witness to the non-primality of 2021.